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BEST EXPONENTS IN MARKOV'S INEQUALITIES

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Abstract. By means of weakly equilibrium Cantor-type sets, solutions of two problems related to polynomial inequalities are presented: the problem by M. Baran et al. about a compact set $K \subset \mathbb{C}$ such that the Markov inequality is not valid on K with the best Markov's exponent, and the problem by L. Frerick et al. concerning compact sets satisfying the local form of Markov's inequality with a given exponent, but not satisfying the global version of Markov's inequality with the same parameter.

1. Two versions of Markov's inequality

The classical Markov inequality $|P'|_{[-1,1]} \leqslant (degP)^2\,|P|_{[-1,1]}$ can be generalized in different ways. We consider two versions that are the most important for applications.

Let $\mathscr{P}_n$ denote the set of all holomorphic polynomials of degree at most n. For any infinite compact set $K \subset \mathbb{C}$ we consider the sequence of *Markov's factors* $M_n(K) = \inf\{M : |P'|_K \leqslant M\,|P|_K,\ P \in \mathscr{P}_n\}$ for $n \in \mathbb{N}$. Here and in what follows, $|\cdot|_K$ denotes the uniform norm on K.

We see that $M_n(K)$ is the norm of the operator of differentiation in the space $(\mathscr{P}_n, |\cdot|_K)$.

In the case of a polynomial growth rate of Markov's factors, we denote $K \in MI(m)$ if there is a constant C such that $M_n(K) \leqslant Cn^m$ for all n. Then the infimum $m(K)$ of all $m > 0$ such that this inequality is valid for some C is called ([2, p. 2786]) the *best Markov's exponent* of K.

We say that the *(global) Markov inequality* is valid on K (or K has *Markov's property*) if $K \in MI(m)$ for some m. Then we write $K \in MI$.

If a compact set K has Markov's property, then the Markov inequality is not necessarily valid on K with the best Markov's exponent. An example of such compact set in the form of a cusp in $\mathbb{C}^N$ for $N \geqslant 2$ was presented by M. Baran, L. Białas-Cież, and B. Milowka in [3, p. 643]. The authors posed the question
[3, p. 638]: is the same true in $\mathbb{C}$?

PROBLEM 1. Give an example of $K \subset \mathbb{C}$ such that the Markov inequality with the exponent $m(K)$ is not valid on K.

In the case of non-polar K, the knowledge about a character of smoothness of the corresponding Green function may help to estimate $M_n(K)$ from above. Let $\hat{K}$ denote

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the polynomial hull of K and $g_{\mathbb{C}\setminus K}$ be the Green function of $\mathbb{C}\setminus\hat{K}$ with pole at infinity (see e.g. [17, p. 108]). If K is regular with respect to the Dirichlet problem then the function $g_{\mathbb{C}\setminus K}$ is continuous throughout $\mathbb{C}$ and we extend it to be zero on $\hat{K}$. The application of the Cauchy formula for P' and the Bernstein-Walsh inequality yields the estimate

$$M_n(K) \leqslant \inf_{\delta} \delta^{-1} \exp[n\cdot\omega(\delta)], \tag{1}$$

where $\omega(\cdot)$ denotes the modulus of continuity of $g_{\mathbb{C}\setminus K}$.

This gives an effective bound of $M_n(K)$ for the cases of temperate growth of $\omega(\cdot)$. For instance, the Hölder continuity of $g_{\mathbb{C}\setminus K}$, which means the existence of constants C,α :

$$g_{\mathbb{C}\setminus K}(z) \leqslant C\,(dist(z,K))^{\alpha} \quad \text{for all } \; z\in\mathbb{C}$$

(then we write $g_{\mathbb{C}\setminus K}\in Lip\;\alpha$) implies $K\in MI(1/\alpha)$. Indeed, given $n\in\mathbb{N}$, the value $\delta = n^{-1/\alpha}$ in (1) gives the result.

In particular, the global Markov inequality is valid on uniformly perfect sets, since the Green function for any uniformly perfect set is Hölder continuous (see e.g. [6, p. 65]). Recall that a compact set K is *uniformly perfect* if it has at least two points and the moduli of annuli in the complement of K which separate K are bounded.

Markov's property is closely related to problems of polynomial approximation and extension of C^∞ functions. W. Pleśniak in [15, T.3.3, p. 111] presented an extension operator for the space $\mathscr{E}(K)$ of Whitney functions on K with Markov's property. This operator is continuous in so-called Jackson topology, which is equivalent to the natural topology of $\mathscr{E}(K)$ if and only if the compact set K satisfies the Markov property. It should be noted that such operator was introduced in [14, p. 285] for a more special family of uniformly polynomially cuspidal compact subsets of $\mathbb{R}^n$.

However, there are sets K without Markov's property, but such that the spaces $\mathscr{E}(K)$ admit the extension operator ([8, p. 31], [9, p. 167], [1, T. 1, p. 38]).

Not only the global version of Markov's inequality, but also its local version is related to the extension problem ([11], [5], [7]).

We say that the *local Markov inequality* with parameter m is valid on $K\subset\mathbb{C}$ (we write $K\in LMI(m)$) if there exist constants $(C_n)_{n=1}^{\infty}$ with $C_n\geqslant 1$ such that for each $P\in\mathscr{P}_n$, each $\varepsilon\in(0,1]$, and each $z\in K$ we have

$$|P'(z)| \leqslant C_n\,\varepsilon^{-m}\,|P|_{K\cap B(z,\varepsilon)}.$$

Here, $B(z,\varepsilon)$ is the closed ball of radius ε centered at z.

Naturally, we write $K\in LMI$ if $K\in LMI(m)$ for some m.

Some authors (cf. [5], p. 854) call the inequality above the weak local Markov inequality, whereas LMI in [13, p. 203] is $LMI(1)$ in our notations. We follow here [7, p. 592], where L. Frerick, E. Jordá, and J. Wengenroth presented a linear tame extension operator for the Whitney space $\mathscr{E}(K)$, provided $K\in LMI(m)$ for some m. In particular L. Frerick et al. posed in [7, p. 602]:

PROBLEM 2. Present $K\in LMI(m)$, not satisfying the global version of Markov's inequality with the same m.

We can answer both questions by means of so-called weakly equilibrium Cantor sets.

For basic definitions and facts of Logarithmic Potential Theory see e.g. [16] and [17], log denotes the natural logarithm.

2. Weakly equilibrium Cantor sets

For the convenience of the reader we repeat the relevant material from [10], thus making our exposition self-contained. Given sequence $\gamma = (\gamma_s)_{s=1}^{\infty}$ with $0 < \gamma_s < 1/4$, we define a sequence of real polynomials

$$P_2(x) = x(x-1) \qquad \text{and} \qquad P_{2^{s+1}} = P_{2^s}(P_{2^s} + r_s)$$

for $s \in \mathbb{N}$ with $r_0 = 1$ and $r_s = \gamma_s r_{s-1}^2$ for $s \geqslant 1$. We consider

$$E_s := \{x \in \mathbb{R} : P_{2^{s+1}}(x) \leqslant 0\} = \cup_{j=1}^{2^s} I_{j,s}.$$

The closed *basic* intervals $I_{j,s}$ of the s-th level are disjoint and $E_{s+1} \subset E_s$. This gives a Cantor type set $K(\gamma) := \cap_{s=0}^{\infty} E_s$.

The sequence of level domains $D_s = \{z \in \mathbb{C} : |P_{2^s}(z) + r_s/2| < r_s/2\}$, $s \in \mathbb{N}$, is a nested family: $\overline{D}_s \searrow K(\gamma)$. Consequently, for the corresponding Robin constants we have $Rob(\overline{D}_s) = 2^{-s} \log \frac{2}{r_s} \nearrow Rob(K(\gamma))$. Thus the set $K(\gamma)$ is polar if and only if $\lim_{s\to\infty} 2^{-s} \log \frac{2}{r_s} = \infty$. If this limit is finite and $z \notin K(\gamma)$, then

$$g_{\mathbb{C}\setminus K(\gamma)}(z) = \lim_{s\to\infty} 2^{-s} \log |P_{2^s}(z)/r_s|.$$

Let $\delta_s = \gamma_1 \gamma_2 \cdots \gamma_s$, so $r_1 r_2 \cdots r_{s-1}\, \delta_s = r_s$. Since $|P'_{2^s}(0)| = r_s/\delta_s$ and $|P_{2^s} + r_s/2|_{K(\gamma)} = r_s/2$, we get for $s \in \mathbb{N}$

$$M_{2^s}(K(\gamma)) \geqslant 2/\delta_s. \tag{2}$$

In what follows we will consider only $K(\gamma)$ satisfying the assumption

$$\gamma_s \leqslant 1/32 \quad \text{for} \quad s \in \mathbb{N}, \tag{3}$$

which provides some additional properties of the sets. In particular, they are *weakly equilibrium* in the following sense.

Let us uniformly distribute the mass 2^{-s} on each $I_{j,s}$ for $1 \leqslant j \leqslant 2^s$. We denote by λ_s the normalized in this sense Lebesgue measure on the set E_s. If the set $K(\gamma)$ is not polar, then λ_s converges in the weak* topology to the equilibrium measure of the set $K(\gamma)$.

The lengths $l_{j,s}$ of the intervals $I_{j,s}$ of the s-th level are not the same, but we can estimate them in terms of the parameter δ_s ([10], (9) and L.6):

$$\delta_s < l_{1,s} < 2\delta_s, \quad \delta_s < l_{j,s} < \delta_s \cdot \exp(16 \sum_{k=1}^{s} \gamma_k) \;\text{ for }\; 2 \leqslant j \leqslant 2^s. \tag{4}$$

If $I_{i,s} \subset I_{j,s-1}$ then ([10], Cor.2)

$$\frac{1}{2}\gamma_s l_{j,s-1} < l_{i,s} < 4\gamma_s l_{j,s-1}. \tag{5}$$

If the set $K(\gamma)$ is not polar, then we can characterize smoothness of $g_{\mathbb{C}\setminus K(\gamma)}$ in terms of the parameter $\rho_s = \sum_{k=s+1}^{\infty} 2^{-k} \log \frac{1}{2\gamma_k}$. It has potential theory meaning: $\rho_s = Rob(K(\gamma)) - Rob(\overline{D}_s)$, so it shows how rapidly the minimal energy for the set $\overline{D}_s$ approximates the energy corresponding to $\mu_{K(\gamma)}$. By T. 5 in [10],

$$\rho_s + 2^{-s}\log\frac{\delta}{\delta_s} < \omega(g_{\mathbb{C}\setminus K(\gamma)}, \delta) < \rho_s + 2^{-s}\log\frac{16\,\delta}{\delta_s} \tag{6}$$

for $\delta_s \leqslant \delta < \delta_{s-1}$. This and (1) provide

EXAMPLE 1. ([10], Ex. 6) Given $m_0 \geqslant 5$, let $\gamma_s = 2^{-m_0}$ for all s. Then $m(K(\gamma)) = m_0$.

In view of the possibility to represent $K(\gamma)$ in terms of a controllable sequence of polynomials, these sets are highly suitable for applications to polynomial inequalities.

3. Markov's exponents for $K(\gamma)$

In the synthesis of Example 1, we estimate $m(K(\gamma))$ for the general case.

LEMMA 1. *Let $(x_k)_{k=1}^{\infty}$ be a sequence of real numbers. Suppose $x_1 + x_2 + \cdots + x_n \leqslant C_0$ with some constant C_0 for all n. Then for $C_1 = C_0 e^{C_0} + e^{-1}$ we have for all n*

$$e^{x_1+x_2+\cdots+x_n}\left(x_{n+1} + \frac{1}{2}x_{n+2} + \cdots + \frac{1}{2^{k-1}}x_{n+k} + \cdots\right) \leqslant C_1.$$

Proof. Given fixed n, let $t_n = x_1 + x_2 + \cdots + x_n$. Then $t_n e^{t_n} \geqslant -e^{-1}$. By assumption, $2^{-i}(x_{n+1} + \cdots + x_{n+i}) \leqslant 2^{-i}(C_0 - t_n)$. Summing these inequalities for $i = 1, 2, \cdots, k-1, k, k$ we get $e^{t_n}(x_{n+1} + \frac{1}{2}x_{n+2} + \cdots + \frac{1}{2^{k-1}}x_{n+k}) \leqslant e^{t_n}(C_0 - t_n) \leqslant C_1$, which is the desired conclusion. □

THEOREM 1. *For the set $K(\gamma)$ with (3) and $m \in \mathbb{R}$, $m \geqslant 5$ the following statements are equivalent.*

(i) $g_{\mathbb{C}\setminus K(\gamma)} \in Lip\,\frac{1}{m}$.

(ii) $K(\gamma) \in MI(m)$.

(iii) There exists a constant C such that $C \cdot \delta_s \geqslant 2^{-ms}$ for all $s \in \mathbb{N}$.

Thus, $m(K(\gamma)) = \inf\{m : \sup_s(\log_2 1/\delta_s - ms) < \infty\}$.

Proof. The implication $(i) \Rightarrow (ii)$ follows (1) and the next argument.

From (2) we have the implication $(ii) \Rightarrow (iii)$.

Let us show that (iii) implies (i). The set $K(\gamma)$, provided (iii), is not polar. Indeed, $1/\gamma_k = \delta_{k-1}/\delta_k \leqslant C 2^{(m-5)k+5}$, by (3) and (iii). Therefore the series that represents ρ_s converges, $Rob(K(\gamma)) < \infty$, and we can use (6).

Suppose $\delta_s \leqslant \delta < \delta_{s-1}$. We proceed to show that $\rho_s \leqslant C_1\,\delta_s^{1/m}$ for some constant C_1. By the substitution $\gamma_k = 2^{-m}\,e^{-mx_k}$ we reduce the proof of this inequality to Lemma 1. Here, $\delta_s = 2^{-ms}\,e^{-m(x_1+x_2+\cdots+x_s)}$ and (iii) provides $x_1+x_2+\cdots+x_s \leqslant C_0 = \frac{1}{m}\log C$. On the other hand, $\log\frac{1}{2\gamma_k} = (m-1)\log 2 + mx_k$, so $\rho_s = (m-1)\,2^{-s}\log 2 + m\,2^{-s-1}(x_{s+1}+\frac{1}{2}x_{s+2}+\cdots)$. Lemma 1 now yields the desired inequality.

It remains to prove that $2^{-s}\,\log\frac{16\,\delta}{\delta_s} \leqslant C_2\,\delta^{1/m}$ for some constant C_2. Fix $\beta \in (0,1]$ such that $\delta = \delta_s^{\beta}\,\delta_{s-1}^{1-\beta}$. Then $\delta = \delta_s T$ with $T = (1/\gamma_s)^{1-\beta} \geqslant 1$.

Applying (iii) we reduce the required inequality to the evident form

$$2^{-s}\log 16 + 2^{-s}\log T \leqslant C_2\,C^{-1/m}\,2^{-s}\,T^{1/m}.$$

By (6), $\omega(g_{\mathbb{C}\setminus K(\gamma)},\delta) < (C_1+C_2)\,\delta^{1/m}$ and $g_{\mathbb{C}\setminus K(\gamma)} \in Lip\,\frac{1}{m}$. $\square$

EXAMPLE 2. (to Problem 1) For each $m_0 \geqslant 5$ there exists $K(\gamma)$ with $m(K(\gamma)) = m_0$ such that $K(\gamma) \notin MI(m_0)$.

Indeed, let $N_k \nearrow \infty$ with $N_0 = 0, N_k = o(k)$ (for example $N_k = \sqrt{k}$). Take $\gamma_k = 2^{-m_0-N_k+N_{k-1}}$. Then $\delta_s = 2^{-s\cdot m_0 - N_s}$. Here the condition $\exists C : C\cdot\delta_s \geqslant 2^{-ms}, \forall s$ is valid for each $m > m_0$ but is not valid for $m = m_0$.

COROLLARY 1. (Compare to T. 3 in [19, p. 721]) *The following are equivalent.*
(I) $g_{\mathbb{C}\setminus K(\gamma)}$ *is Hölder continuous.*
(II) $K(\gamma)$ *has Markov's property.*
(III) There exists $a > 0$ *such that* $\delta_s \geqslant a^s$ *for all* $s \in \mathbb{N}$.

The condition (iii) of Theorem 1 does not imply that $K(\gamma)$ is uniformly perfect. By Theorem 3 in [10], the set $K(\gamma)$ is uniformly perfect if and only if $\inf_s \gamma_s > 0$. Let us construct Markov's set $K(\gamma)$ even though it is not α-perfect for any α given beforehand.

4. α-perfect sets

Given $\alpha \geqslant 1$, a compact set $K \subset \mathbb{C}$ is a perfect set of the class α ([18, p. 74]) if there are constants $C \geqslant 1, \delta > 0$ such that for any $y \in K$ one can find a sequence $(x_j)_{j=1}^{\infty} \subset K$ such that $|y - x_j| \downarrow 0$, $|y - x_1| \geqslant \delta$ and $C\cdot|y - x_{j+1}| \geqslant |y - x_j|^{\alpha}$ for any $j \in \mathbb{N}$. In this case we will write $K \in (\alpha)$.

Thus, $K \in (1)$ means that K is uniformly perfect.

If $K \in MI(m)$ then K is α-perfect for $\alpha > 2m$. This was proved by Jonsson in [12, T. 3, p. 96].

On the other hand, in unpublished preprint by B.Uzun and the author the following examples were suggested: $K_1 \in (\alpha)$ for any $\alpha > 1$ without Markov's property, and, given α, the set $K_2 \notin (\alpha)$ with Markov's property. For the convenience of the reader we repeat these examples here and, then, will give their Cantor versions.

EXAMPLE 3. Let $K_1 = \{0\} \cup \cup_{k=2}^{\infty} I_k$, where $I_k = [a_k, b_k] = [c_k - \delta_k, c_k + \delta_k]$ with $b_k = \frac{1}{k!}$, $a_k = b_k - b_{k+1}$. Clearly, K_1 is α-perfect set for any $\alpha > 1$. We claim that K_1 does not satisfy the Markov property.

We use the Chebyshev polynomials $T_N(x) = \cos(N \cdot \arccos x)$ for $|x| \leqslant 1$, $N \in \mathbb{Z}_+$. Given k, let T_{Nk} denote the Chebyshev polynomial corresponding to the interval I_k, that is $T_{Nk}(x) = T_N(\frac{x-c_k}{\delta_k})$. The well-known representation $T_N(x) = \frac{1}{2}[(x+\sqrt{x^2-1})^N + (x-\sqrt{x^2-1})^N]$ for $|x| > 1$ implies

$$(\Delta/\delta_k)^N < |T_{Nk}(c_k \pm \Delta)| < (2\Delta/\delta_k)^N \text{ for } \Delta > \delta_k. \tag{7}$$

For fixed large n, let us consider the polynomial $P(x) = x \cdot \prod_{k=2}^n \beta_k T_{N_k k}(x)$, with $\beta_k = T_{N_k k}^{-1}(0)$. We say that P is a "nearly Chebyshev" polynomial for K_1, if degrees $(N_k)_{k=2}^n$ are chosen in a such way that the maximal values of $|P|$ on the intervals $(I_k)_{k=2}^n$ are approximately the same. It is difficult to find the best possible values of these degrees. For our aim it is enough to take $N_k = [\log^{n-k} n]$, where $[a]$ denotes the greatest integer in a.

Clearly, $deg\, P < \log^n n$ and $P'(0) = 1$. We will show that for each $x \in K_1$

$$|P(x)| \leqslant b_n. \tag{8}$$

It will follow the absence of Markov's property of K_1, since $n! \leqslant C \cdot (\log n)^{n \cdot m}$ is a contradiction for fixed C, m and $n \to \infty$.

Let us fix $x \in K_1$. If $0 \leqslant x \leqslant b_k$ then $|\beta_k T_{N_k k}(x)| \leqslant 1$. For this reason (8) is valid for $x \leqslant b_n$.

Fix j with $2 \leqslant j \leqslant n-1$ such that $x \in I_j$. Then

$$|P(x)| \leqslant b_j |\beta_j| \cdot \prod_{k=j+1}^{n} |\beta_k \cdot T_{N_k k}(b_j)|,$$

since all other terms of the product are less than 1 on I_j. From (7) we have

$$|\beta_k \cdot T_{N_k k}(b_j)| < \left(\frac{2(b_j - c_k)}{c_k}\right)^{N_k} < \left(\frac{2b_j}{c_k}\right)^{N_k} = \left(\frac{4(k+1)!}{j!(2k+1)}\right)^{N_k} < \left(\frac{4\, k!}{j!}\right)^{N_k}.$$

Therefore, $\prod_{k=j+1}^n |\beta_k \cdot T_{N_k k}(b_j)| < 4^{N_{j+1}+\cdots+N_n} \prod_{k=j+1}^n (k!/j!)^{N_k}$. The last product here is $\prod_{k=j+1}^n k^{N_k+\cdots+N_n}$.

On the other hand, by (7), $|T_{N_j j}(0)| > (c_j/\delta_j)^{N_j} = (2j+1)^{N_j}$. From this,

$$|P(x)| \leqslant \frac{1}{j!} (2j+1)^{-N_j} 4^{N_{j+1}+\cdots+N_n} \prod_{k=j+1}^{n} k^{N_k+\cdots+N_n}.$$

Since $N_k + \cdots + N_n < (\log n - 1)^{-1} \log^{n-k+1} n$, it suffices to show that

$$\log \frac{n!}{j!} + \frac{\log^{n-j} n}{\log n - 1} \left[\log 4 + \sum_{k=j+1}^{n} \frac{\log k}{\log^{k-j-1} n}\right] < N_j \log(2j+1).$$

The expression in brackets does not exceed $\log 4 + 2\log(j+1)$ and $\frac{\log 4 + 2\log(j+1)}{\log n - 1} < \frac{1}{2} \log(2j+1)$ for large n. On the other hand, $N_j > \log^{n-j} n - 1$. Thus, the desired inequality (8) can be reduced to the form

$$\log \frac{n!}{j!} < \log(2j+1) \cdot \left[\frac{1}{2} \log^{n-j} n - 1\right],$$

which is easy to check by considering the following cases:

if $2 \leqslant j \leqslant n - \log n$ then $\log \frac{n!}{j!} < \log n! < n \cdot \log n < \log 5 \cdot \left[\frac{1}{2} \log^{\log n} n - 1\right]$,

if $n - \log n < j \leqslant n-2$ then $\log \frac{n!}{j!} < \log^2 n < \log n \cdot \left[\frac{1}{2} \log^2 n - 1\right]$, and

if $j = n-1$ then $\log n < \log(2n-1) \cdot \left[\frac{1}{2} \log n - 1\right]$.

The inequalities above are valid for large enough n, one can take $n > e^7$.

Before the next example, let us find the exact coefficient in the well-known inequality which states that $g_{\mathbb{C}\setminus I} \in Lip\ 1/2$ for an interval I.

LEMMA 2. *Suppose $I = [-\delta, \delta]$. Then $g_{\mathbb{C}\setminus I}(z) \leqslant \sqrt{2\Delta/\delta}$ for $z \in \mathbb{C}$ with $dist(z,I) \leqslant \Delta$. The coefficient $\sqrt{2/\delta}$ is sharp.*

Proof. Since the level curves of the function $g_{\mathbb{C}\setminus I}$ are ellipses with focii $\pm\delta$, it attains its maximal value, among all z with $dist(z,I) = \Delta$, at the real points:

$$\max\{g_{\mathbb{C}\setminus I}(z) : dist(z,I) = \Delta\} = g_{\mathbb{C}\setminus I}(\delta + \Delta) = \log(1 + t + \sqrt{2t + t^2})$$

with $t = \Delta/\delta$. Let us consider the function $C(t) = t^{-1/2} \log(1 + t + \sqrt{2t+t^2})$ for $t > 0$. We have $C(+0) = \sqrt{2}$ and $C'(t) = (2t\sqrt{t})^{-1} \left[2t/\sqrt{2t+t^2} - \log(1+t+\sqrt{2t+t^2})\right]$. The expression in brackets is negative, as is easy to check. Therefore the function C decreases, which gives the desired inequality. □

EXAMPLE 4. For arbitrary $N > 2$, let $K_2 = \{0\} \cup \cup_{k=0}^{\infty} I_k$, where $I_k = [a_k, b_k]$ with $b_k = e^{-N^k}$ and $a_k = b_k^2$. As above, let $|I_k| = 2\,\delta_k$. Here, the set K_2 is α-perfect if and only if $\alpha \geqslant \frac{N}{2}$. We claim that the function $g_{\mathbb{C}\setminus K_2}$ is Hölder continuous, so K_2 has Markov's property.

By the Wiener criterion (see e.g. [16, T. 5.4.1, p. 146]), the set K_2 is regular, so $g_{\mathbb{C}\setminus K_2}$ vanishes on K_2 and is continuous throughout $\mathbb{C}$.

Fix $z \notin K_2$ and n with $a_{n+1} \leqslant \Delta = dist(z, K_2) < a_n$. Fix $\zeta \in K_2$ with $\Delta = |z - \zeta|$. If $\zeta \in I_k$ for $k \leqslant n$ then the monotonicity of the Green function and Lemma 2 imply $g_{\mathbb{C}\setminus K_2}(z) \leqslant g_{\mathbb{C}\setminus I_k}(z) \leqslant \sqrt{2\Delta/\delta_k}$. Here, $2\Delta/\delta_k < 2\,a_n/\delta_n = 4\,b_n^2/(b_n - b_n^2) < 4\,b_n/(1 - b_0) < 8\,b_n = 8\,a_{n+1}^{1/2N}$. Hence, $g_{\mathbb{C}\setminus K_2}(z) \leqslant 2\sqrt{2}\Delta^{1/4N}$.

If $\zeta \leqslant b_{n+1}$ then $dist(z, I_n) \leqslant |z - a_n| \leqslant |z - \zeta| + |\zeta - a_n| \leqslant \Delta + a_n < 2\,a_n$. Arguing as above, we see that $g_{\mathbb{C}\setminus K_2}(z) \leqslant g_{\mathbb{C}\setminus I_n}(z) \leqslant 4\Delta^{1/4N}$. Therefore, $g_{\mathbb{C}\setminus K_2} \in Lip\,\frac{1}{4N}$ and $K_2 \in MI(4N)$.

We turn to Cantor versions of the examples above. The following proposition generalizes Theorem 3 in [10].

PROPOSITION 1. *The set $K(\gamma)$ is α-perfect if and only if there exists a constant C such that $C\cdot\gamma_{s+1} \geqslant l_{j,s}^{\alpha-1}$ for all $s\in\mathbb{N}$ and $j\leqslant 2^s$.*

Proof. Each basic interval $I_{j,s}$ contains two adjacent subintervals of the next level, let $I_{i,s+1}$ and $I_{k,s+1}$. By definition of α-perfect sets, $K(\gamma)\in(\alpha)$ if and only if for some constant C we have $C\cdot l_{i,s+1}\geqslant (l_{j,s}-l_{k,s+1})^{\alpha}$, which implies the desired characterization, by (5) and (3). □

EXAMPLE 5. Let $\gamma_s = 1/s$ for $s\geqslant 32$ and $\gamma_s = 1/32$ for $s<32$. Then $K(\gamma)$ is α-perfect for all $\alpha>1$, but $K(\gamma)\notin MI$. By the discussion above, this is impossible for uniformly perfect sets.

Indeed, for large s we have $\delta_s = C_0\,(s!)^{-1}$ with $C_0 = 32!/32^{32}$. Hence, by (4) and (3), $l_{j,s}\leqslant C_0\,e^{s/2}(s!)^{-1}$, which provides $K(\gamma)\in(\alpha)$ for $\alpha>1$. On the other hand, by Corollary 1, the supposition $K(\gamma)\in MI$ will give a contradiction $1/s!\geqslant a^s$ for fixed $a>0$ and all s.

In order to give a $K(\gamma)$-version of Example 4 we use an irregular case, when γ_s is the same for all s, except a subsequence where the values γ_{s_k} are rather small. Given sequences $(s_k)_{k=1}^{\infty}$ of natural numbers and $(\varepsilon_k)_{k=1}^{\infty}$ of positive numbers, let $\gamma_s = \gamma_0\leqslant 1/32$ for $s\neq s_k$ and $\gamma_{s_k}=\gamma_0\,\varepsilon_k$ otherwise. Then, clearly, $\delta_s = \gamma_0^s\,\varepsilon_1\,\varepsilon_2\cdots\varepsilon_k$ for $s_k\leqslant s<s_{k+1}$. In the following example let us take $\gamma_0 = 1/32$.

EXAMPLE 6. Given $\alpha\geqslant 1$, as large as desired, there exist sequences $(s_k)_{k=1}^{\infty}$ and $(\varepsilon_k)_{k=1}^{\infty}$ such that the corresponding set $K_\alpha(\gamma)$ is not α-perfect. At the same time Markov's inequality is valid on $K_\alpha(\gamma)$.

We will choose the sequences that provide $\gamma_{s_k}\cdot\delta_{s_{k-1}}^{1-\alpha}\to 0$ as $k\to\infty$. Then, by (4) and Proposition 1, we have $K_\alpha(\gamma)\notin(\alpha)$. Thus, we need

$$\varepsilon_k\cdot(2^{-5s_k}\,\varepsilon_1\cdots\varepsilon_{k-1})^{1-\alpha}\to 0 \quad\text{as}\quad k\to\infty. \tag{9}$$

On the other hand, if $m\geqslant 5$ and $\delta_{s_k}2^{ms_k}\geqslant 1$ for all k, then $\delta_s 2^{ms}\geqslant 1$ for all s. Indeed, for $s_k\leqslant s<s_{k+1}$ we have $\delta_s=\delta_{s_k}2^{-5(s-s_k)}$ and $\delta_s 2^{ms}=\delta_{s_k}2^{ms_k}2^{(m-5)(s-s_k)}\geqslant 1$. Then, by Theorem 1, $K_\alpha(\gamma)\in MI(m)$ provided

$$\delta_{s_k}2^{ms_k} = 2^{(m-5)\,s_k}\,\varepsilon_1\,\varepsilon_2\cdots\varepsilon_k\geqslant 1 \ \text{ for all }\ k. \tag{10}$$

Let us take $\varepsilon_k = 2^{-(6\alpha)^k}$ for all k, $s_1=1$ and $s_k=[(6\alpha)^k/(6\alpha-1)]$ for $k\geqslant 2$. where $[a]$ denotes the greatest integer in a. A trivial verification shows validity of (9) and (10) for $m\geqslant 6(\alpha+1)$.

Taking into account T. 3 in [12] and the examples above, one can pose

PROBLEM. Given $m\geqslant 1$ find α_m which is the greatest lower bound of α with the property: if $K\in MI(m)$ then K is α-perfect.

5. Local Markov's inequality

Theorem 1 above gives the best Markov's exponents only for the sets $K(\gamma)$. In general, the problem of finding these exponents is rather difficult. For example, we do not know $m(K_0)$ for the classical Cantor ternary set K_0, which is Markov, by [4]. In contrast to this, the characterization of exact classes $LMI(m)$ can be presented for a wide family of Cantor-type sets.

Let $K = \bigcap_{s=0}^{\infty} E_s$, where $E_0 = [0,1]$, $E_s = \cup_{j=1}^{2^s} I_{j,s}$ with $|I_{j,s}| = l_{j,s}$ and E_{s+1} is obtained by deleting an open subinterval of the length $h_{j,s}$ from each $I_{j,s}$ for $1 \leqslant j \leqslant 2^s$. Thus, $l_{j,s} = l_{2j-1,s+1} + h_{j,s} + l_{2j,s+1}$.

Suppose K satisfies the following two mild conditions:

(a) $\exists \sigma_0 \in (0,1) : h_{j,s} \geqslant \sigma_0 \cdot l_{j,s}$ for all $s \in \mathbb{Z}_+$ and $j \leqslant 2^s$

(b) $\forall s \in \mathbb{Z}_+ \ \exists H_s : l_{i,s+q} \leqslant H_s \cdot l_{j,s+q}$ for all $q \in \mathbb{N}$ and for all $I_{i,s+q}, I_{j,s+q} \subset I_{k,q}$.

Clearly, the condition (b) is valid for geometrically symmetric Cantor-type sets, when the lengths of basic intervals of the same level are equal in length. By (5), it is also valid for $K(\gamma)$ with $H_s = 8^s$.

By definition, $K \in LMI(m)$ if and only if

$$C_n := \sup_{P \in \mathscr{P}_n} \sup_{\varepsilon > 0} \sup_{x \in K} \varepsilon^m \frac{|P'(x)|}{|P|_{K \cap B(x,\varepsilon)}} < \infty \quad \text{for all} \quad n \in \mathbb{N}. \tag{11}$$

THEOREM 2. *Suppose a Cantor-type set K satisfies (a),(b) and $m > 0$ is fixed. Then $K \in LMI(m)$ if and only if for each $s \in \mathbb{N}$ there exists $C = C(s,m)$ such that $l_{j,q}^m \leqslant C \cdot l_{i,s+q}$ for all $q \in \mathbb{N}$, $j \leqslant 2^q$ and $I_{i,s+q} \subset I_{j,q}$.*

Proof. Assume that $K \in LMI(m)$, but, by contradiction, there exists s with

$$\sup_{q} \sup_{j \leqslant 2^q} \sup_{I_{i,s+q} \subset I_{j,q}} l_{j,q}^m l_{i,s+q}^{-1} = \infty. \tag{12}$$

Given s, let us take $C_{2^{s-1}}$ from (11) and $N > H_s C_{2^{s-1}} (2/\sigma_0)^m e^{s/2\sigma_0}$, where H_s and σ_0 are defined by (a) and (b). For this N, by (12), we can choose $q, j \leqslant 2^q$ and i with $I_{i,s+q} \subset I_{j,q}$ such that $l_{j,q}^m l_{i,s+q}^{-1} > N$. By I we denote the interval of $q+1$-st level containing $I_{i,s+q}$. There are 2^{s-1} points on I, let $(z_k)_{k=1}^{2^{s-1}}$, that are the endpoints of basic intervals of $q+s-1$-st level. We consider the polynomial $Q(x) = \prod_{k=1}^{2^{s-1}} (x - z_k)$, so each subinterval $I_{k,s+q}$ of I contains one point z_k. Fix $y \in K \cap I$ such that $|Q|_{K \cap I} = |Q(y)|$ and i_0 with $y \in I_{i_0,s+q}$. Let $z_{i_0} \in I_{i_0,s+q}$. Then $|Q'(z_{i_0})| = \prod_{k=1,k\neq i_0}^{2^{s-1}} |z_{i_0} - z_k|$. On the other hand, $|y - z_{i_0}| \leqslant l_{i_0,s+q}$ and $|Q(y)| \leqslant l_{i_0,s+q} \prod_{k=1,k\neq i_0}^{2^{s-1}} |z_{i_0} - z_k| \cdot \beta$ with $\beta = \prod_{k=1,k\neq i_0}^{2^{s-1}} |1 + \frac{y - z_{i_0}}{z_{i_0} - z_k}|$. We fix the chain of intervals containing $y : I_{i_0,s+q} \subset I_{i_1,s+q-1} \subset \cdots \subset I = I_{i_{s-1},q+1}$. Taking into account only z_k for which $\frac{y - z_{i_0}}{z_{i_0} - z_k} > 0$, we obtain, as in Lemma 11 in [10], $\log \beta < \sum \frac{y - z_{i_0}}{z_{i_0} - z_k} \leqslant l_{i_0,s+q} (h_{i_1,s+q-1}^{-1} + 2h_{i_2,s+q-2}^{-1} + \cdots 2^{s-2} h_{i_{s-1},q+1}^{-1})$. Here, by (a), $h_{i_1,s+q-1} \geqslant \sigma_0 \cdot l_{i_1,s+q-1} > 2\sigma_0 \cdot l_{i_0,s+q}, \cdots, h_{i_{s-1},q+1} > 2^{s-1} \sigma_0 \cdot l_{i_0,s+q}$. Therefore, $\log \beta < (s-1)/2\sigma_0$ and $|Q|_{K \cap I} < l_{i_0,s+q} |Q'(z_{i_0})| e^{s/2\sigma_0}$.

Since the local Markov inequality with parameter m is valid on K, we can apply it to $Q \in \mathscr{P}_{2^{s-1}}, x = z_{i_0}$ and $\varepsilon = \sigma_0/2 \cdot l_{j,q}$. Then $B(x,\varepsilon) \cap K \subset I \cap K$, which gives $|Q'(z_{i_0})| \leqslant C_{2^{s-1}} (2/\sigma_0)^m l_{j,q}^{-m} |Q|_{K \cap I}$ and $l_{j,q}^m < C_{2^{s-1}} (2/\sigma_0)^m e^{s/2\sigma_0} l_{i_0,s+q}$. By (b), this yields $l_{j,q}^m < N l_{i,s+q}$, a contradiction.

We proceed to prove $K \in LMI(m)$ provided the given geometric condition. Since the value C_n in (11) increases with n, we can consider only n from some subsequence, let $n = 2^s - 1$ for $s \in \mathbb{N}$. Fix $P \in \mathscr{P}_n$, $x_0 \in K$ and $\varepsilon \in (0,1)$. Given x_0 defines a chain of basic intervals $I_{1,0} \supset I_{j_1,1} \supset \cdots \supset I_{j_q,q} \supset \cdots$ containing x_0. Let us fix q with $l_{j_q,q} \leqslant \varepsilon < l_{j_{q-1},q-1}$ and denote $I_{j_q,q}$ by I. Then $B(x_0,\varepsilon) \cap K \supset I \cap K$. The interval I contains 2^s points, let $(z_k)_{k=1}^{2^s}$, that are the endpoints of basic intervals of $q+s-1$-st level. We interpolate P at these points, so $P(x) = \sum_{k=1}^{2^s} P(z_k)\, L_k(x)$ for $L_k(x) = \frac{Q(x)}{(x-z_k)\,Q'(z_k)}$ with $Q(x) = \prod_{k=1}^{2^s}(x - z_k)$. Our goal is to show that

$$|L_k'(x_0)| \leqslant N\,\varepsilon^{-m} \quad \text{for} \quad 1 \leqslant k \leqslant 2^s, \tag{13}$$

where N depends only on s and m. Provided (13) we get the desired result, since $|P'(x_0)| \leqslant 2^s\,|P|_{I\cap K} \max_{1\leqslant k\leqslant 2^s} |L_k'(x_0)| \leqslant N\,2^s\,\varepsilon^{-m}\,|P|_{B(x_0,\varepsilon)\cap K}$.

Clearly, $|L_k'(x_0)| \leqslant |Q'(z_k)|^{-1} \prod_{i=1,i\neq k}^{2^s} |x_0 - z_i| \sum_{i=1,i\neq k}^{2^s} |x_0 - z_i|^{-1}$. The interval I covers 2^s subintervals of $s+q$-th level, each of them contains one point z_k. Let us first consider the case $z_k \in I_{j_{s+q},s+q}$, so z_k and x_0 are on the same subinterval of $s+q$-th level. Here, $\prod_{i=1,i\neq k}^{2^s} |x_0 - z_i| \leqslant \tau := l_{j_{s+q-1},s+q-1}\, l_{j_{s+q-2},s+q-2}^2 \cdots l_{j_q,q}^{2^{s-1}}$. By (a),

$$|Q'(z_k)| = \prod_{i=1,i\neq k}^{2^s} |z_k - z_i| \geqslant l_{j_{s+q-1},s+q-1}\, h_{j_{s+q-2},s+q-2}^2 \cdots h_{j_q,q}^{2^{s-1}} \geqslant \sigma_0^{2^s-2}\,\tau,$$

and

$$\sum_{i=1,i\neq k}^{2^s} |x_0 - z_i|^{-1} \leqslant \sigma_0^{-1}\,[l_{j_{s+q-1},s+q-1}^{-1} + 2\,l_{j_{s+q-2},s+q-2}^{-1} + \cdots + 2^{s-1}\,l_{j_q,q}^{-1}]$$
$$< 2^s \sigma_0^{-1}\, l_{j_{s+q-1},s+q-1}^{-1}.$$

Therefore, $|L_k'(x_0)| \leqslant 2^s \sigma_0^{1-2^s}\, l_{j_{s+q-1},s+q-1}^{-1}$. By condition,

$$\varepsilon^m < l_{j_{q-1},q-1}^m \leqslant C(s,m)\, l_{j_{s+q-1},s+q-1},$$

which gives (13) for the first case.

Now assume that z_k and x_0 belong to different subinterval of $s+q$-th level. Fix r such that $z_r \in I_{j_{s+q},s+q}$ and the chain $z_k \in I_{i_{s+q},s+q} \subset \cdots \subset I_{i_q,q} = I$. Here, by (a) and (b), $|Q'(z_k)| \geqslant l_{i_{s+q-1},s+q-1}\, h_{i_{s+q-2},s+q-2}^2 \cdots h_{i_q,q}^{2^{s-1}} \geqslant \sigma_0^{2^s-2}\, H_s^{-2^s+1}\,\tau$ with the same τ as above. To deal with the rest, we single the term $|x_0 - z_r|$ out: $\prod_{i\neq k} |x_0 - z_i| \sum_{i\neq k} |x_0 - z_i|^{-1} = \prod_{i\neq k, i\neq r} |x_0 - z_i| \cdot [1 + |x_0 - z_r| \sum_{i\neq k,i\neq r} |x_0 - z_i|^{-1}]$. Now, $\prod_{i\neq k,i\neq r} |x_0 - z_i| = |x_0 - z_k|^{-1} \prod_{i\neq r} |x_0 - z_i| \leqslant |x_0 - z_k|^{-1}\,\tau$, as before, and

$$[\cdots] \leqslant 1 + l_{j_{s+q},s+q}\,\sigma_0^{-1}\,(l_{j_{s+q-1},s+q-1}^{-1} + \cdots + 2^{s-1}\, l_{j_q,q}^{-1}) < 1 + (2^s - 1)/\sigma_0 < 2^s/\sigma_0.$$

Combining these inequalities yields $|L_k'(x_0)| \leqslant (H_s/\sigma_0)^{2^s}\, 2^s\, |x_0 - z_k|^{-1}$, which also gives (13) for the same reason as in the first case, since $|x_0 - z_k| > h_{j_{s+q-1},s+q-1} \geqslant \sigma_0\, l_{j_{s+q-1},s+q-1}$. $\square$

PROPOSITION 2. *Suppose $(\gamma_s)_{s=1}^{\infty}$ satisfies (3) and $m \geqslant 1$. If $K(\gamma) \in LMI(m)$ then for each $s \in \mathbb{N}$ there exists $C = C(s,m)$ such that $\delta_q^m \leqslant C \cdot \delta_{s+q}$ for all $q \in \mathbb{N}$. For two model cases $\sum_{s=1}^{\infty} \gamma_s < \infty$ and $\gamma_s = \gamma_1$ for all s the inverse implication is valid as well.*

Proof. Suppose $K(\gamma) \in LMI(m)$. The values $i = j = 1$ in Theorem 2 and applying both inequalities in (4) yield the desired conclusion.

In the case $\sum_{s=1}^{\infty} \gamma_s < \infty$, by (4), Theorem 2 and the given geometric condition imply that $K(\gamma) \in LMI(m)$.

If $\gamma_s = \gamma_1$ for all s then the condition on (δ_q) is trivially valid for all $m \geqslant 1$. On the other hand, here the set $K(\gamma)$ is uniformly perfect. Then, by J. Lithner ([13, Prop. 5.1, p. 209]), $K \in LMI(1)$, so $K \in LMI(1)$ for all $m \geqslant 1$. $\square$

EXAMPLE 7. (to Problem 2) Let us take $\gamma_s = \gamma_1 \leqslant \frac{1}{32}$ for all s. Then, by Theorem 1, the global version of Markov's inequality is valid only for $m \geqslant -\frac{\log \gamma_1}{\log 2}$, whereas the local form of Markov's inequality is valid for all $m \geqslant 1$.

The sets $K(\gamma)$ are not convenient to distinguish classes $LMI(m)$ for different m. It is better to use for this aim geometrically symmetric Cantor-type sets. By means of a sequence $A = (A_s)_{s=1}^{\infty}$ we define the set $K(A)$, as in the beginning of this section, with $0 < l_1 < 1/2$ and $|I_{j,s}| = l_s = l_1^{A_s}$ for all $j \leqslant 2^s$. The values $(A_s)_{s=1}^{\infty}$ with $\liminf_s (A_{s+1} - A_s) > \log 2 / \log l_1^{-1}$ provide the condition (a).

PROPOSITION 3. *Suppose $K(A)$ satisfies the condition (a) and $m \geqslant 1$. Then*

1) the set $K(A)$ is α-perfect if and only if there exists a constant C such that $A_{s+1} - \alpha A_s \leqslant C$ for all $s \in \mathbb{N}$;

2) $K(A) \in LMI(m)$ if and only if for each $s \in \mathbb{N}$ there exists $C = C(s,m)$ such that $A_{s+q} \leqslant mA_q + C$ for all $q \in \mathbb{N}$.

Proof. Indeed, the first statement follows from the definition of α-perfect sets. The second characterization is a corollary of Theorem 2. $\square$

V. Totik proved in [19, T. 3, p. 721] that $K(A)$ has Markov's property if and only if the sequence $(A_s/s)_{s=1}^{\infty}$ is bounded. As we mentioned above, the problem of characterization of exact classes $MI(m)$ is far from the solution. Here, by means of irregular sequences $(A_s)_{s=1}^{\infty}$, we distinguish classes $LMI(m)$ and show that, in general, the local Markov inequality is not valid with the best local Markov exponent.

EXAMPLE 8. For any $m \geqslant 1$ there exists a set $K(A) \notin LMI(m)$ with $K(A) \in LMI(m+\varepsilon)$ for each $\varepsilon > 0$.

Let us take $l_1 = 1/3$. Then $A_{q+1} = A_q + 1$ means that $3\,l_{q+1} = l_q$. Suppose $A_{q+1} = A_q + 1$ for $q \neq q_n$ and $A_{q_n+1} = mA_{q_n} + n$ for $n \in \mathbb{N}$. Here $(q_n)_{n=1}^{\infty}$ is a sequence of natural numbers with $q_{n+1} - q_n \uparrow \infty$ and $q_n/n \to \infty$ as $n \to \infty$.

Since $\sup_q (A_{q+1} - mA_q) = \infty$, we have $K(A) \notin LMI(m)$, by Proposition 3.

On the other hand, suppose $\varepsilon > 0$ and $s \in \mathbb{N}$ are fixed. Let us show that $\sup_q [A_{q+s} - (m+\varepsilon)A_q]$ is finite. Fix n_0 such that $\varepsilon q_{n-1} > n$ and $q_{n+1} - q_n > s$ for $n > n_0$. Since $A_q \geqslant q$ for all q, we have $\varepsilon A_q > \varepsilon A_{q_{n-1}} > n$ for $q_{n-1} < q \leqslant q_n$. If $q > q_{n_0}$ then there is at most one value q_n between q and $q+s$. If $[q, q+s] \cap (q_n)_{n=1}^{\infty} = \emptyset$

then $A_{q+s} = A_q + s \leqslant (m+\varepsilon)A_q + C$ for $C = s$. Otherwise there exists q_n with $q \leqslant q_n \leqslant q+s$, let $q_n = q+k$ with $k \leqslant s$. Then $A_{q+k} = A_q + k$, $A_{q+k+1} = m(A_q+k)+n$ and $A_{q+s} = m(A_q+k)+n+s-1 < (m+\varepsilon)A_q + C$ for $C = ms$. Therefore, the limit above does not exceed $C(s,m) = \max\{ms, \max_{q \leqslant q_{n_0}}[A_{q+s} - (m+\varepsilon)A_q]\}$.

The last two examples are related to comparison of classes $MI(m)$ and $LMI(m)$.

EXAMPLE 9. Let $A_q = q \cdot \log q$ for $q \geqslant 2$. Then $K(A) \in LMI(m)$ for each $m > 1$, but $K(A)$ does not satisfy the Markov property.

Indeed, the global Markov inequality is not valid on $K(A)$ as the sequence $(A_q/q)_{s=1}^{\infty}$ is not bounded. On the other hand, given $m > 1$ and $s \in \mathbb{N}$, let $q_0 = \frac{s}{m-1}$. Then the value $C(s,m) = \max\{s\log 4, \max_{q \leqslant q_0}[A_{q+s} - mA_q]\}$ provides the inequality $A_{s+q} \leqslant mA_q + C(s,m)$, as easy to check.

EXAMPLE 10. For each m, as large as desired, there exist Markov's set $K = K_m(A)$ with $K(A) \notin LMI(m)$.

Given m, fix $l_1 < 2^{-m}$. Let $\delta_0 = \log 2/\log l_1^{-1}$, so $\delta_0 < 1/m$. Fix δ with $\delta_0 < \delta < 1/m$. Suppose a sequence $(q_n)_{n=1}^{\infty}$ of natural numbers is given. Let $A_{q_n} = q_n$ for all n and $A_{q+1} = A_q + \delta$ for $q \neq q_n - 1$. Thus, $A_q = q_n + (q - q_n)\,\delta$ for $q_n \leqslant q < q_{n+1}$. The condition $A_{q+1} = A_q + \delta$ means that $l_{q+1} = l_q\, l_1^{\delta} < l_q/2$, so the set K is well-defined. Since K satisfies (a), we can use Proposition 3.

Here $A_q \leqslant q$ for all q. By Totik's characterization, K has Markov's property. But $K \notin LMI(m)$ for a suitable choice of $(q_n)_{n=1}^{\infty}$. Otherwise, for $s = 1$ there is a constant $C = C(1,m)$ such that $A_{q+1} \leqslant mA_q + C$ for all q. The value $q = q_{n+1} - 1$ gives $q_{n+1} \leqslant m[q_n + (q_{n+1} - 1 - q_n)\cdot\delta] + C$, which is a contradiction for large n in the case of fast growing sequence $(q_n)_{n=1}^{\infty}$, for example $q_n = 2^{n^2}$.

The set K above belongs to $LMI(m_1)$ with $m_1 = 1/\delta$. In more general, if $K(A) \in MI$ then $K(A) \in LMI(m)$ with $m = C_0 \log l_1^{-1}/\log 2$, where $C_0 = \sup_q A_q/q$. Indeed, $l_q = l_1^{A_q} < 2^{-q}$, so $A_q > q \cdot \log 2/\log l_1^{-1}$ for all q. Therefore, $A_{s+q} \leqslant C_0(s+q) < mA_q + C_0\, s$.

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